A Simple Proof of Gauss Markov Theorem

Starting point: The Sample Mean is BLUE

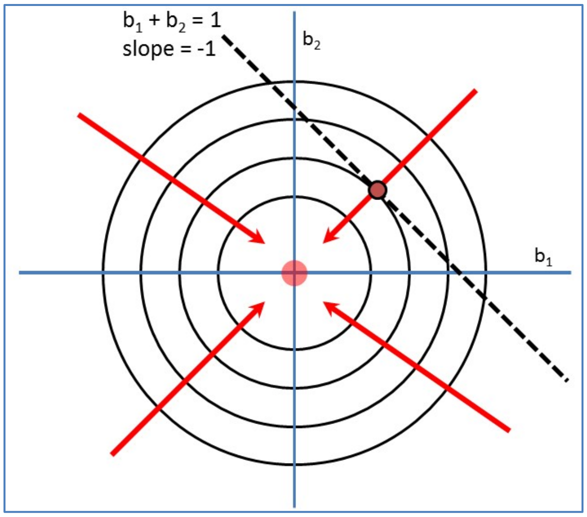
1) Recall the analysis of the Sample Mean estimator…
2) Linear unbiased estimators (LUEs)

   $M = \beta_1 Y_1 + \beta_2 Y_2 + ... + \beta_n Y_n$, where $\sum_{i=1}^{n} \beta_i = 1$

   The constraint is required for unbiasedness.
3) To find the Best Linear Unbiased Estimator (BLUE):

   min $Var(\sum \beta_i Y_i) = \sigma^2 \sum \beta_i^2$

   subject to $\sum_{i=1}^{n} \beta_i = 1$.
4) Solution: $\beta_i^* = \frac{1}{n}$.
5) So the Sample Mean is a BLUE Estimator.



Turning to Gauss-Markov: Estimating the slope parameter in an SLR model

6) We want to estimate the slope parameter, β_1, of the linear model (DGM): $Y = \beta_0 + \beta_1 X + U$. Assume SLR.1-SLR.5 and consider the following general linear estimator (since we are conditioning on the x_i's, the estimator will be linear in the Y_i's): $B_1 = b_0 + \sum b_i Y_i$.
7) ***Unbiased:*** Since $E\left[b_0 + \sum b_i Y_i\right] = b_0 + \sum b_i\left(\beta_0 + \beta_1 x_i\right) = b_0 + \beta_0 \sum b_i + \beta_1 \sum b_i x_i \equiv \beta_1$, for a LUE, we require:
   - $b_0 = 0$,
   - ***Constraint I***: $\sum b_i = 0$ and
   - ***Constraint II***: $\sum b_i x_i = 1$.[1]
8) ***Variance:*** Given that $b_0 = 0$, the variance of the estimator will be: $Var\left[\sum b_i Y_i\right] = \sigma^2 \sum b_i^2$. And so the optimization problem becomes:

$$min \;\; \sigma^2 \sum b_i^2 \;\; s.t. \;\; \sum b_i = 0 \text{ and } \sum b_i x_i = 1$$

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[1] Note that if SLR.3 is violated and there is no variation in the x's, then the two constraints cannot be jointly satisfed so long as $\bar{x} \neq 0$.

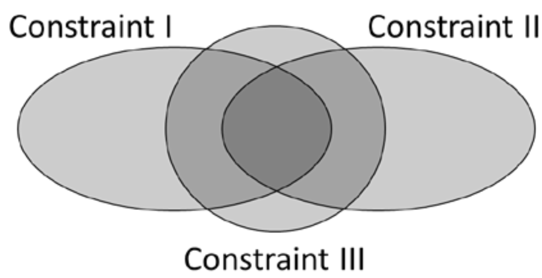
9) Collapsing the constraints:

   a) Constraints I and II imply that $1 = \sum b_i x_i = \sum b_i x_i - \bar{x}\sum b_i$, since $\sum b_i = 0$, and so any $\{b_i\}$ satisfying I and II, must also satisfy:

   - **Constraint III**: $\sum b_i \left(x_i - \bar{x}\right) = \sum b_i d_i = 1 \ldots$ where $d_i = \left(x_i - \bar{x}\right),\ i = 1, \ldots, n$

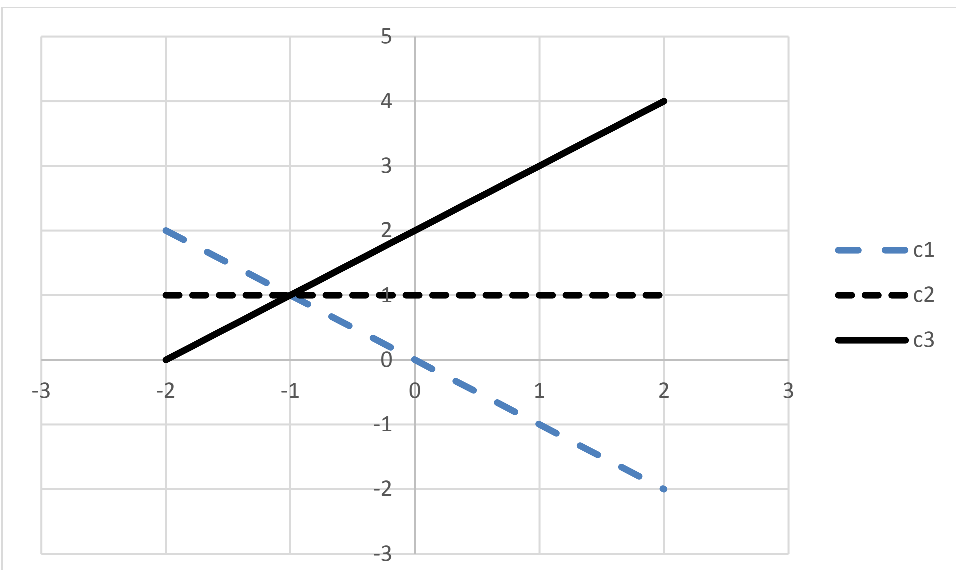
10) The trick below is that rather than minimizing $\sigma^2 \sum b_i^2$ subject to Constraints I and II, we instead minimize subject to Constraint III, which includes all $\{b_i\}$ satisfying I and II, and perhaps other $\{b_i\}$, and then show that the optimal b's, $\{b_i^*\}$, also satisfy Constraints I and II.

11) The following diagram illustrates the approach:



12) Example: Consider two x's, where $x_1 = 0$ and $x_2 = 1$. Then the three constraints are:

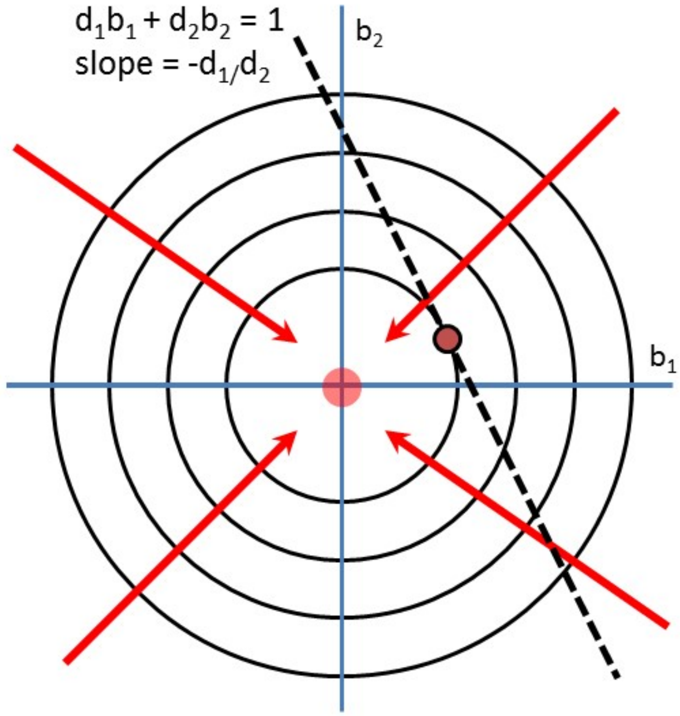
   - Constraint I (c1): $b_1 + b_2 = 0$,
   - Constraint II (c2): $b_2 = 1$, and
   - Constraint III (c3): $.5b_2 - .5b_1 = 1$.



Note that the $\{b_i\}$ satisfying constraints I and II ($b_1 = -1, b_2 = 1$) also satisfies constraint III.

13) Accordingly, the Constraint III BLUE optimization problem becomes:

$$min \ \ \sigma^2 \sum b_i^2 \ \ s.t. \sum \left(x_i - \overline{x} \right) b_i = \sum d_i b_i = 1 .$$



14) At the optimum: $b_i^* = \frac{d_i}{\sum_{j=1}^{n} d_j^2} = \frac{\left(x_i - \overline{x} \right)}{\sum_j \left(x_j - \overline{x} \right)^2} \ \ i = 1, ..., n, .$[2]

15) And so, we have ***OLS!***:

$$B_1 = \sum_i \left(\frac{\left(x_i - \overline{x} \right)}{\sum_j \left(x_j - \overline{x} \right)^2} \right) Y_i = \frac{\sum_i \left(x_i - \overline{x} \right) Y_i}{\sum_j \left(x_j - \overline{x} \right)^2} = \frac{\sum_i \left(x_i - \overline{x} \right) \left(Y_i - \overline{Y} \right)}{\sum_j \left(x_j - \overline{x} \right)^2}$$

[2] See the Appendix for a proof.

16) Checking that both constraints I and II are satisfied:

a) Constraint I is satisfied: $\sum b_i x_i = \dfrac{\sum (x_i - \overline{x}) x_i}{\sum_j (x_j - \overline{x})^2} = 1$ since for the denominator, we have $\sum_j (x_j - \overline{x})^2 = \sum_j (x_j - \overline{x})(x_j - \overline{x}) = \sum_j (x_j - \overline{x}) x_j$.

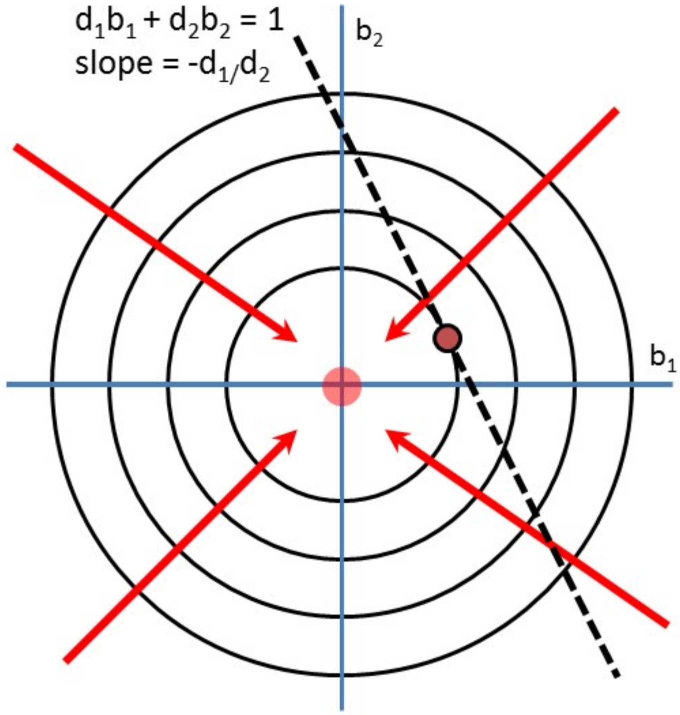
b) Constraint II is also satisfied: $\sum b_i = \dfrac{\sum (x_i - \overline{x})}{\sum_j (x_j - \overline{x})^2} = 0$ because the numerator is 0.

c) Constraints satisfied!

OLS is BLUE!

Appendix: Proof of the Gauss-Markov Theorem

17) Consider the constrained optimization problem above (since it doesn't affect anything, let's drop σ^2): min $\sum_{i=1}^{n}\beta_i^2 = 1$ subject to $\sum_{i=1}^{n} d_i\beta_i = 1$.



a) As you saw in the ***Review of Estimation***, we can incorporate the *constraint* into the *objective function*. The constraint requires that $\sum_{i=1}^{n} d_i\beta_i = 1$, or put differently:

$d_n\beta_n = 1 - \sum_{i=1}^{n-1} d_i\beta_i$, or $\beta_n = \frac{1}{d_n}\left[1 - \sum_{i=1}^{n-1} d_i\beta_i\right]$.

b) Incorporating the constraint in the objective function, we have a new (unconstrained) optimization problem:

$$\min \sum_{i=1}^{n-1}\beta_i^2 + \beta_n^2 = \left\{\sum_{i=1}^{n-1}\beta_i^2 + \frac{1}{d_n^2}\left[1 - \sum_{i=1}^{n-1} d_i\beta_i\right]^2\right\}.$$

c) As before, we can solve this with n-1 FOCs (First Order Conditions):

For $j = 1, \ldots, n-1$: $\frac{\partial}{\partial \beta_j}\left\{\sum_{i=1}^{n-1}\beta_i^2 + \beta_n^2\right\} = \left\{2\beta_j + \frac{2}{d_n^2}\left[1-\sum_{i=1}^{n-1}d_i\beta_i\right](-d_j)\right\} = 0$

d) But then $\beta_j^* = \frac{d_j}{d_n^2}\left[1-\sum_{i=1}^{n-1}d_i\beta_i^*\right] = \frac{d_j}{d_n^2}d_n\beta_n^* = \frac{d_j}{d_n}\beta_n^*$, and so $\frac{\beta_j^*}{d_j} = \frac{\beta_i^*}{d_i} = \frac{\beta_n^*}{d_n}$, for any *i* and *j*.

e) Simplifying things:

i) $\sum_{j=1}^{n-1}d_j\beta_j^* = \frac{\beta_n^*}{d_n}\sum_{j=1}^{n-1}d_j^2 = 1 - d_n\beta_n^*$, or … $\beta_n^*\sum_{j=1}^{n-1}d_j^2 = d_n - d_n^2\beta_n^*$

ii) $\beta_n^*\sum_{j=1}^{n-1}d_j^2 + d_n^2\beta_n^* = d_n$, or … $\beta_n^*\sum_{j=1}^{n}d_j^2 = d_n$, or … $\beta_n^* = \frac{d_n}{\sum_{j=1}^{n}d_j^2}$

iii) So $\frac{\beta_j^*}{d_j} = \frac{\beta_i^*}{d_i} = \frac{\beta_n^*}{d_n} = \frac{1}{\sum_{j=1}^{n}d_j^2}$ … and so $\beta_i^* = \frac{d_i}{\sum_{j=1}^{n}d_j^2}$ $i = 1, ..., n$.

f) Or put differently: At the (constrained) optimum, $d_i\beta_i^* = \frac{d_i^2}{\sum_{j=1}^{n}d_j^2}$ $i = 1, ..., n$, and the

$d_i\beta_i^*$'s are non-negative weights that sum to one.